

Component Evolution in General Random Intersection Graphs

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Abstract

Random intersection graphs (RIGs) are an important random structure with applications in social networks, epidemic networks, blog readership, and wireless sensor networks. RIGs can be interpreted as a model for large randomly formed non-metric data sets. We analyze the component evolution in general RIGs, and give conditions on existence and uniqueness of the giant component. Our techniques generalize existing methods for analysis of component evolution: we analyze survival and extinction properties of a dependent, inhomogeneous Galton-Watson branching process on general RIGs. Our analysis relies on bounding the branching processes and inherits the fundamental concepts of the study of component evolution in Erdős-Rényi graphs. The major challenge comes from the underlying structure of RIGs, which involves its both the set of nodes and the set of attributes, as well as the set of different probabilities among the nodes and attributes.

Keywords: Random graphs, branching processes, probabilistic methods, random generation of combinatorial structures, stochastic processes in relation with random discrete structures.

1 Introduction

Bipartite graphs, consisting of two sets of nodes with edges only connecting nodes in opposite sets, are a natural representation for many networks. A well-known example is a collaboration graph, where the two sets might be scientists and research papers, or actors and movies [25, 16]. Social networks can often be cast as bipartite graphs since they are built from sets of individuals connected to sets of attributes, such as membership of a club or organization, work colleagues, or fans of the same sports team. Simulations of epidemic spread in human populations are often performed on networks constructed from bipartite graphs of people and the locations they visit during a typical day [11]. Bipartite structure, of course, is hardly limited to social networks. The relation between nodes and keys in secure wireless communication, for examples, forms a bipartite network [6]. In general, bipartite graphs are well suited to the problem of classifying objects, where each object has a set of properties [10]. However, modeling such classification networks remains a challenge. The well-studied Erdős-Rényi model, $G_{n,p}$, successfully used for average-case analysis of algorithm performance, does not satisfactorily represent many randomly formed social or collaboration networks. For example, $G_{n,p}$ does not capture the typical scale-free degree distribution of many real-world networks [3]. More realistic degree distributions can be achieved by the configuration model [18] or expected degree model [7], but even those fail to capture common properties of social networks such as the high number of triangles (or cliques) and strong degree-degree correlation [17, 1].

The most straightforward way of remedying these problems is to characterize each of the bipartite sets separately. One step in this direction is an extension of the configuration model that specifies degrees in both sets [14]. Another related approach is that of random intersection graphs (RIG), first introduced in [24, 15]. Any undirected

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graph can be represented as an intersection graph [9]. The simplest version is the “uniform” RIG, $G(n, m, p)$, containing a set of n nodes and a set of m attributes, where any given node-attribute pair contains an edge with a fixed probability p , independently of other pairs. Two nodes in the graph are taken to be connected if and only if they are both connected to at least one common element in the attribute set. In our work, we study the more general RIG, $G(n, m, \mathbf{p})$ [20, 19], where the node-attribute edge probabilities are not given by a uniform value p but rather by a set $\mathbf{p} = \{p_w\}_{w \in W}$: a node is attached to the attribute w , with probability p_w . This general model has only recently been developed and only a few results have obtained, such as expander properties, cover time, and the existence and efficient construction of large independent sets [20, 19, 21].

In this paper, we analyze the evolution of components in general RIGs. Related results have previously been obtained for the uniform RIG [4], and for two uniform cases of the RIG model where a specific overlap threshold controls the connectivity of the nodes, were analyzed in [6]. Our main contribution is a generalization of the component evolution on a general RIG. We provide stochastic bounds, by analyzing the stopping time of the branching process on general RIG, where the history of the process is directly dictated by the structure of the general RIG. The major challenge comes from the underlying structure of RIGs, which involves both the set of nodes and the set of attributes, as well as the set of different probabilities $\mathbf{p} = \{p_w\}_{w \in W}$.

2 Model and previous work

In this paper, we will consider the general intersection graph $G(n, m, \mathbf{p})$, introduced in [20, 19], with a set of probabilities $\mathbf{p} = \{p_w\}_{w \in W}$, where $p_w \in (0, 1)$. We now formally define the model.

Model. There are two sets: the set of nodes $V = \{1, 2, \dots, n\}$ and the set of attributes $W = \{1, 2, \dots, m\}$. For a given set of probabilities $\mathbf{p} = \{p_w\}_{w \in W}$, independently over all $(v, w) \in V \times W$ let

$$A_{v,w} := \text{Bernoulli}(p_w). \quad (1)$$

Every node $v \in V$ is assigned a random set of attributes $W(v) \subseteq W$

$$W(v) := \{w \subseteq W \mid A_{v,w} = 1\}. \quad (2)$$

The set of edges in V is defined such that two different nodes $v_i, v_j \in V$ are connected if and only if

$$|W(v_i) \cap W(v_j)| \geq s, \quad (3)$$

for a given integer $s \geq 1$.

In our analysis, p_w are not necessarily the same as in [4, 6]¹, and for simplicity we fix $s = 1$.

The component evolution of the uniform model $G(n, m, p)$ was analyzed by Behrisch in [4], for the case when the scaling of nodes and attributes is $m = n^\alpha$, with $\alpha \neq 1$ and $p^2 m = c/n$. Theorem 1 in [4] states that the size of the largest component $\mathcal{N}(G(n, m, p))$ in RIG satisfies (i) $\mathcal{N}(G(n, m, p)) \leq \frac{9}{(1-c^2)} \log n$, for $\alpha > 1, c < 1$, (ii) $\mathcal{N}(G(n, m, p)) = (1 + o(1))(1 - \rho)n$, for $\alpha > 1, c > 1$, (iii) $\mathcal{N}(G(n, m, p)) \leq \frac{10\sqrt{c}}{(1-c^2)} \sqrt{\frac{n}{m}} \log m$, for $\alpha < 1, c < 1$, (iv) $\mathcal{N}(G(n, m, p)) = (1 + o(1))(1 - \rho)\sqrt{cmn}$, for $\alpha < 1, c > 1$, where ρ is the solution in $(0, 1)$ of the equation $\rho = \exp(c(\rho - 1))$.

The component evolution for the case $s \geq 1$ in the relation $|W(u) \cap W(v)| \geq s$ is considered in [6], where the following two RIG models are analyzed: (1) $G_s(n, m, d)$ model, where $\mathbb{P}[W(v) = A] = \binom{m}{d}^{-1}$ for all $A \subseteq W$ on d elements, for a given d ; (2) $G'_s(n, m, p)$ model, where $\mathbb{P}[W(v) = A] = p^{|A|}(1 - p)^{m-|A|}$ for all $A \subseteq W$. In light of results of [4], it has been shown in [6], that for $d = d(n), p = p(n), m = m(n), n = o(m)$, where

¹Note that p_w 's do not sum up to 1. Moreover, we can eliminate the cases $p_w = 0$ and $p_w = 1$. These two cases respectively correspond when none or all nodes v are attached to the attribute w .

s is a fixed integer, and $d^{2s} \sim cm^s s! / n$, the largest component in $G_s(n, m, d)$ satisfies: (i) $\mathcal{N}(G_s(n, m, d)) \leq \frac{9}{(1-c^2)} \log n$, for $c < 1$, (ii) $\mathcal{N}(G_s(n, m, d)) = (1 + o(1))(1 - \rho)n$, for $c > 1$, in the case when $n \log n = o(m)$ for $s = 1$ and $n = o(m^{s/(2s-1)})$ for $s \leq 2$. The same results for the giant component in $G_s(n, m, p)$ still hold for the case when $p^{2s} = cs! / m^s n$ and $n = o(m^{s/(2s-1)})$, see [6].

Both $G_s(n, m, d)$ and $G'_s(n, m, p)$ are special cases of a more general class studied in [13], where the number of attributes of each node is assigned randomly as in the bipartite configuration model. That is, for a given probability distribution (P_0, P_1, \dots, P_m) , we have $\mathbb{P}[|W(v)| = k] = P_k$ for all $0 \leq k \leq m$, and moreover given the size k , all of the sets $W(v)$ are equally probable, that is for any $A \subseteq W$, $\mathbb{P}[W(v) = A : |W(v)| = k] = \binom{m}{k}^{-1}$. That is, we see that $G_s(n, m, d)$ is equivalent to the model of [13] with the delta-distribution, where the probability of the d -th coordinate is 1, while $G'_s(n, m, d)$ is equivalent to the model of [13] with the $\text{Bin}(m, p)$ distribution. To complete the picture of previous work, in [8], it was shown that when $n = m$ a set of probabilities $\mathbf{p} = \{p_w\}_{w \in W}$ can be chosen to tune the degree and clustering coefficient of the graph.

3 Mathematical preliminaries

In this paper, we analyze the component evolution of the general RIG structure. As we have already mentioned, the major challenge comes from the underlying structure of RIGs, which involves both the set of nodes and the set of attributes, as well as the set of different probabilities $\mathbf{p} = \{p_w\}_{w \in W}$.

Moreover, the edges in RIG are not independent. Hence, a RIG cannot be treated as an Erdős-Rényi random graph $G_{n, \hat{p}}$, with the edge probability $\hat{p} = 1 - \prod_{w \in W} (1 - p_w^2)$. However, in [12], the authors provide the comparison among $G_{n, \hat{p}}$ and $G(n, m, p)$, showing that for $m = n^\alpha$ and $\alpha > 6$, these two classes of graphs have asymptotically the same properties. In [23], Rybarczyk has recently shown the equivalence of sharp threshold functions among $G_{n, \hat{p}}$ and $G_{n, m, p}$, when $m \geq n^3$. In this work, we do not impose any constraints among n and m , and we develop methods for the analysis of branching processes on RIGs, since the existing methods for the analysis of branching processes on $G_{n, p}$ do not apply.

We now briefly state the edge dependence. Consider three distinct nodes v_i, v_j, v_k from V . Conditionally on the set $W(v_k)$, by the definition (2), the sets $W(v_i) \cap W(v_k)$ and $W(v_j) \cap W(v_k)$ are mutually independent, which implies conditional independence of the events $\{v_i \sim v_k \mid W(v_k)\}, \{v_j \sim v_k \mid W(v_k)\}$, that is,

$$\mathbb{P}[v_i \sim v_k, v_j \sim v_k \mid W(v_k)] = \mathbb{P}[v_i \sim v_k \mid W(v_k)] \mathbb{P}[v_j \sim v_k \mid W(v_k)]. \quad (4)$$

However, the latter does not imply independence of the events $\{v_i \sim v_k\}$ and $\{v_j \sim v_k\}$ since in general

$$\begin{aligned} \mathbb{P}[v_i \sim v_k, v_j \sim v_k] &= \mathbb{E}[\mathbb{P}[v_i \sim v_k, v_j \sim v_k \mid W(v_k)]] \\ &= \mathbb{E}[\mathbb{P}[v_i \sim v_k \mid W(v_k)] \mathbb{P}[v_j \sim v_k \mid W(v_k)]] \\ &\neq \mathbb{P}[v_i \sim v_k] \mathbb{P}[v_j \sim v_k]. \end{aligned} \quad (5)$$

Furthermore, the conditional pairwise independence (4) does not extend to three or more nodes. Indeed, conditionally on the set $W(v_k)$, the sets $W(v_i) \cap W(v_j)$, $W(v_i) \cap W(v_k)$, and $W(v_j) \cap W(v_k)$ are not mutually independent, and hence neither are the events $\{v_i \sim v_j\}, \{v_i \sim v_k\}$, and $\{v_j \sim v_k\}$, that is,

$$\mathbb{P}[v_i \sim v_j, v_i \sim v_k, v_j \sim v_k \mid W(v_k)] \neq \mathbb{P}[v_i \sim v_j \mid W(v_k)] \mathbb{P}[v_i \sim v_k \mid W(v_k)] \mathbb{P}[v_j \sim v_k \mid W(v_k)]. \quad (6)$$

We now provide two identities, which we will use throughout this paper. For any $w \in W$, let $q_w := 1 - p_w$, and define $\prod_{\alpha \in \emptyset} q_\alpha = 1$.

Claim 1 For any node $u \in V$ and given set $A \subseteq W$,

$$\mathbb{P}[W(u) \cap A = \emptyset \mid A] = \prod_{\alpha \in A} (1 - p_\alpha) = \prod_{\alpha \in A} q_\alpha. \quad (7)$$

Proof Write

$$\mathbb{P}[W(u) \cap A = \emptyset | A] = \mathbb{P}[\forall \alpha \in A, \alpha \notin W(u) | A] = \prod_{\alpha \in A} \mathbb{P}[\alpha \notin W(u)] = \prod_{\alpha \in A} (1 - p_\alpha) = \prod_{\alpha \in A} q_\alpha,$$

which is the desired expression. ■

Claim 2 For any node $u \in V$, and given sets $A \subseteq B \subseteq W$,

$$\mathbb{P}[W(u) \cap A = \emptyset, W(u) \cap B \neq \emptyset | A, B] = \left(\prod_{\alpha \in A} q_\alpha \right) \left(1 - \prod_{\alpha \in B \setminus A} q_\alpha \right) = \prod_{\alpha \in A} q_\alpha - \prod_{\beta \in B} q_\beta.$$

Proof The sets A and $B \setminus A$ are disjoint. The result follows from (7). ■

4 Auxiliary process on general random intersection graphs

Our analysis for the emergence of a giant component is inspired by the approach described in [2]. The difficulty in analyzing the evolution of the stochastic process defined by equations (1), (2), and (3) resides in the fact that we need, at least in principle, to keep track of the temporal evolution of the sets of nodes and attributes being explored. This results in a process that is not Markovian.

We construct an auxiliary process, which starts at an arbitrary node $v_0 \in V$, and reaches zero for the first time in a number of steps equal to the size of the component containing v_0 . The process is algorithmically defined as follows.

Auxiliary Process. Let us denote by V_t the cumulative set of nodes *visited* by time t , which we initialize to $V_0 = \{v_0\}$, and set $W(v_0) = \{v \neq v_0 : W(v) \cap W(v_0) \neq \emptyset\}$. Starting with $Y_0 = 1$, the process evolves as follows: For $t = 1, 2, 3, \dots, n-1$ and $Y_t > 0$, pick a node v_t uniformly at random from the set $V \setminus V_{t-1}$ and update the set of visited nodes $V_t = V_{t-1} \cup \{v_t\}$. Denote by $W(v_t) = \{w \in W \mid A_{v_t, w} = 1\}$ the set of features associated to node v_t , and define

$$Y_t = \left| \left\{ v \in V \setminus V_t \mid W(v) \cap \bigcup_{\tau=0}^t W(v_\tau) \neq \emptyset \right\} \right|.$$

The random variable Y_t counts the number of nodes outside the set of visited nodes V_t that are connected to V_t . Following [2], we call Y_t the number of *alive* nodes at time t . We note that we do not need to keep track of the actual list of neighbors of V_t

$$\left\{ v \in V \setminus V_t \mid W(v) \cap \bigcup_{\tau=0}^t W(v_\tau) \neq \emptyset \right\}, \quad (8)$$

as in [2], because every node in $V \setminus V_t$ is equally likely to belong to the set (8). As a result, each time we need a random node from (8), we pick a node uniformly at random from $V \setminus V_t$.

To understand why this process is useful, notice that by time t , we know that the size of the component containing v_0 is at least as large as the number of visited nodes V_t plus the number Y_t of neighbors of V_t not yet visited. Once the number Y_t of neighbors connected to V_t but not yet visited drops to zero, the size of V_t is equal to the size of the component containing v_0 . We formalize this last statement by introducing the stopping time

$$T(v_0) = \inf\{t > 0 : Y_t = 0\}, \quad (9)$$

whose value is $|C(v_0)|$.

Finally, our analysis of that process requires us to keep track of the history of the feature sets uncovered by the process

$$\mathcal{H}_t = \{W(v_0), W(v_1), \dots, W(v_t)\}. \quad (10)$$

4.1 Process description in terms of random variable Y_t

As in [6], we denote the cumulative feature set associated to the sequence of nodes v_0, \dots, v_t from the auxiliary process by

$$W_{[t]} := \cup_{\tau=0}^t W(v_\tau). \quad (11)$$

We will characterize the process $\{Y_t\}_{t \geq 0}$ in terms of the number Z_t of newly discovered neighbors to V_t . The latter is directly related to the increment, defined by of the process Y_t

$$Z_t = Y_t - Y_{t-1} + 1, \quad (12)$$

where the term $+1$ reflects the fact that one node, Y_{t-1} decreases by one when the node v_t becomes a visited node at time t . The events that any given node, which is neither visited nor alive, becomes *alive* at time t are conditionally independent given the history \mathcal{H}_t , since each event involves a different subsets of the indicator random variables $\{A_{v,w}\}$. In light of Claim 2, the conditional probability that a node u becomes alive at time t is

$$\begin{aligned} r_t &:= \mathbb{P}[u \sim v_t, u \not\sim v_{t-1}, u \not\sim v_{t-2}, \dots, u \not\sim v_0 | \mathcal{H}_t] \\ &= \mathbb{P}[W(u) \cap W(v_t) \neq \emptyset, W(u) \cap W_{[t-1]} = \emptyset | \mathcal{H}_t] \\ &= \mathbb{P}[W(u) \cap W(v_t) \neq \emptyset, W(u) \cap W_{[t-1]} = \emptyset | W(v_t), W_{[t-1]}] \\ &= \prod_{\alpha \in W_{[t-1]}} q_\alpha - \prod_{\beta \in W_{[t]}} q_\beta \\ &= \phi_{t-1} - \phi_t, \end{aligned} \quad (13)$$

where we set $\phi_t := \prod_{\alpha \in W_{[t]}} q_\alpha$, and use the convention $W_{[-1]} = W(\emptyset) \equiv \emptyset$ and $\phi_{-1} \equiv 1$. Observe that the probability (13) does not depend on u . Hence the number of new alive nodes at time t is, conditionally on the history \mathcal{H}_t , a Binomial distributed random variable with parameters r_t and

$$N_t = n - t - Y_t. \quad (14)$$

Formally,

$$Z_{t+1} | \mathcal{H}_t \sim \text{Bin}(N_t, r_t). \quad (15)$$

This allows us to describe the distribution of Y_t in the next lemma.

Lemma 3 *For times $t \geq 1$, the number of alive nodes satisfies*

$$Y_t | \mathcal{H}_{t-1} \sim \text{Bin}\left(n - 1, 1 - \prod_{\tau=0}^{t-1} (1 - r_\tau)\right) - t + 1. \quad (16)$$

The proof of this lemma requires us to establish the following result first.

Lemma 4 *Let random variables Λ_1, Λ_2 satisfy: $\Lambda_1 \sim \text{Bin}(m, \nu_1)$ and Λ_2 given $\Lambda_1 \sim \text{Bin}(\Lambda_1, \nu_2)$. Then marginally $\Lambda_2 \sim \text{Bin}(m, \nu_1 \nu_2)$ and $\Lambda_1 - \Lambda_2 \sim \text{Bin}(m, \nu_1(1 - \nu_2))$.*

Proof Let U_1, \dots, U_m and V_1, \dots, V_m be i.i.d. Uniform(0, 1) random variables. Writing

$$\Lambda_1 \stackrel{d}{=} \sum_{j=1}^m \mathbb{I}(U_j \leq \nu_1) \quad \text{and} \quad \Lambda_2 | \Lambda_1 \stackrel{d}{=} \sum_{k: U_k \leq \nu_1} \mathbb{I}(V_k \leq \nu_2),$$

we have that

$$\Lambda_2 \stackrel{d}{=} \sum_{k=1}^m \mathbb{I}(U_k \leq \nu_1) \mathbb{I}(V_k \leq \nu_2) \stackrel{d}{=} \sum_{k=1}^m \mathbb{I}(U_k \leq \nu_1 \nu_2),$$

from which the conclusion follows. ■

Proof (Proof of Lemma 3) We prove the assertion on the Lemma by induction in t . For $t = 0$, $Y_0 = 1$ and $t = 1$, $Y_1 = Z_1 \sim \text{Bin}(n-1, r_0)$. Hence, the Lemma is true for $t = 1$ and $t = 0$. Assume that the assertion is true for some $t \geq 1$,

$$Y_t | \mathcal{H}_{t-1} \sim \text{Bin}\left(n-1, 1 - \prod_{\tau=0}^{t-1} (1-r_\tau)\right) - t + 1. \quad (17)$$

From (15), we have $Z_{t+1} | \mathcal{H}_t \sim \text{Bin}(N_t, r_t) = \text{Bin}(n-t-Y_t, r_t)$. Now, from (12) and Lemma 4, it follows

$$Y_{t+1} | \mathcal{H}_t \sim \text{Bin}\left(n-1, 1 - \prod_{\tau=0}^t (1-r_\tau)\right) - t. \quad (18)$$

Hence, by mathematical induction, the Lemma holds for any $t \geq 0$. ■

4.2 Expectation and variance of ϕ_t

The history \mathcal{H}_t embodies the evolution of how the features are discovered over time. It is insightful to recast that history in terms of the discovery times Γ_w of each feature in W . Given any sequence of nodes v_0, v_1, v_2, \dots , the probability that a given feature w is first discovered at time $t < n$ is

$$\begin{aligned} \mathbb{P}[\Gamma_w = t] &= \mathbb{P}[A_{v_t, w} = 1, A_{v_{t-1}, w} = 0, \dots, A_{v_0, w} = 0] \\ &= p_w (1 - p_w)^t. \end{aligned}$$

If a feature w is not discovered by time $n-1$, we set $\Gamma_w = \infty$ and note that

$$\mathbb{P}[\Gamma_w = \infty] = (1 - p_w)^n.$$

From the independence of the random variables $A_{v, w}$, it follows that the discovery times $\{\Gamma_w : w \in W\}$ are independent. We now focus on describing the distribution of $\phi_t = \prod_{\alpha \in W_{[t]}} q_\alpha$. For $t \geq 0$, we have

$$\phi_t = \prod_{\alpha \in W_{[t]}} q_\alpha = \prod_{j=0}^t \prod_{\alpha \in s(v_j) \setminus S[j-1]} q_\alpha \stackrel{d}{=} \prod_{j=0}^t \prod_{w \in W} q_w^{\mathbb{I}(\Gamma_w = j)} = \prod_{w \in W} q_w^{\mathbb{I}(\Gamma_w \leq t)}. \quad (19)$$

Using the fact that for a $B \sim \text{Bernoulli}(r)$, the expectation $\mathbb{E}[a^B] = 1 - (1-a)r$, we can easily calculate the expectation of ϕ_t

$$\begin{aligned} \mathbb{E}[\phi_t] &= \mathbb{E}\left[\prod_{w \in W} q_w^{\mathbb{I}(\Gamma_w \leq t)}\right] = \prod_{w \in W} \left(1 - (1 - q_w) \mathbb{P}[\Gamma_w \leq t]\right) \\ &= \prod_{w \in W} \left(1 - (1 - q_w)(1 - q_w^{t+1})\right). \end{aligned} \quad (20)$$

The concentration of ϕ_0 will be crucial for the analysis of the supercritical regime, Subsection 5.2. Hence, we here provide $\mathbb{E}[\phi_0]$ and $\mathbb{E}[\phi_0^2]$. From (20) it follows

$$\mathbb{E}[\phi_0] = \prod_{w \in W} (1 - p_w^2) = 1 - \sum_{w \in W} p_w^2 + \phi\left(\sum_{w \in W} p_w^2\right). \quad (21)$$

Moreover, from (19) it follows

$$\begin{aligned} \mathbb{E}[\phi_0^2] &= \mathbb{E}\left[\prod_{w \in W} q_w^{2\mathbb{I}(\Gamma_w \leq 0)}\right] = \prod_{w \in W} \left(1 - (1 - q_w^2) \mathbb{P}[\Gamma_w = 0]\right) = \prod_{w \in W} \left(1 - (1 - q_w^2)p_w\right) \\ &= \prod_{w \in W} \left(1 - 2p_w^2 + p_w^3\right) = 1 - 2 \sum_{w \in W} p_w^2 + \phi\left(\sum_{w \in W} p_w^2\right). \end{aligned} \quad (22)$$

5 Giant component

With the process $\{Y_t\}_{t \geq 0}$ defined in the previous section, we analyze both the subcritical and supercritical regime of our random intersection graph by adapting the percolation based techniques to analyze Erdős-Rényi random graphs [2]. The technical difficulty in analyzing that stopping time rests in the fact that the distribution of Y_t depends on the history of the process, dictated by the structure of the general RIG. In the next two subsections, we will give conditions on non-existence, that is, on existence and uniqueness of the giant component in general RIGs.

5.1 Subcritical regime

Theorem 5 *Let*

$$\sum_{w \in W} p_w^3 = O(1/n^2) \quad \text{and} \quad p_w = O(1/n) \text{ for all } w.$$

For any positive constant $c < 1$, if $\sum_{w \in W} p_w^2 \leq c/n$, then all components in a general random intersection graph $G(n, m, \mathbf{p})$ are of order $O(\log n)$, with high probability².

Proof We generalize the techniques used in the proof for the sub-critical case in $G_{n,p}$ presented in [2]. Let $T(v_0)$ be the stopping time define in (9), for the process starting at node v_0 and note that $T(v_0) = |C(v_0)|$. We will bound the size of the largest component, and prove that under the conditions of the theorem, all components are of order $O(\log n)$, **whp**.

For all $t \geq 0$,

$$\begin{aligned} \mathbb{P}[T(v_0) > t] &= \mathbb{E}[\mathbb{P}[T(v_0) > t \mid \mathcal{H}_t]] \leq \mathbb{E}[\mathbb{P}[Y_t > 0 \mid \mathcal{H}_t]] \\ &= \mathbb{E}\left[\mathbb{P}\left[\text{Bin}(n-1, 1 - \prod_{\tau=0}^{t-1} (1-r_\tau)) \geq t \mid \mathcal{H}_t\right]\right]. \end{aligned} \quad (23)$$

Bounding from above, which can easily be proven by induction in t for $r_\tau \in [0, 1]$, we have

$$1 - \prod_{\tau=0}^{t-1} (1-r_\tau) \leq \sum_{\tau=0}^{t-1} r_\tau = \sum_{\tau=0}^{t-1} (\phi_{\tau+1} - \phi_\tau) = 1 - \phi_t. \quad (24)$$

By using stochastic ordering of the Binomial distribution, both in n and in $\sum_{\tau=0}^{t-1} r_\tau$, and for any positive constant ν , which is to be specified later, it follows

$$\begin{aligned} \mathbb{P}[T(v_0) > t \mid \mathcal{H}_t] &\leq \mathbb{P}\left[\text{Bin}(n, \sum_{\tau=0}^{t-1} r_\tau) \geq t \mid \mathcal{H}_t\right] = \mathbb{P}[\text{Bin}(n, 1 - \phi_t) \geq (1-\nu)t \mid \mathcal{H}_t] \\ &= \mathbb{P}[\text{Bin}(n, 1 - \phi_t) \geq t \mid 1 - \phi_t < (1-\nu)t/n \cap \mathcal{H}_t] \mathbb{P}[1 - \phi_t < (1-\nu)t/n \mid \mathcal{H}_t] \\ &\quad + \mathbb{P}[\text{Bin}(n, 1 - \phi_t) \geq t \mid 1 - \phi_t \geq (1-\nu)t/n \cap \mathcal{H}_t] \mathbb{P}[1 - \phi_t \geq (1-\nu)t/n \mid \mathcal{H}_t] \\ &\leq \mathbb{P}[\text{Bin}(n, 1 - \phi_t) \geq t \mid 1 - \phi_t < (1-\nu)t/n \cap \mathcal{H}_t] \\ &\quad + \mathbb{P}[1 - \phi_t \geq (1-\nu)t/n \mid \mathcal{H}_t]. \end{aligned} \quad (25)$$

Furthermore, using the fact that the event $\{1 - \phi_t < (1-\nu)t/n\}$ is \mathcal{H}_t -measurable, together with the stochastic ordering of the binomial distribution, we obtain

$$\mathbb{P}[\text{Bin}(n, 1 - \phi_t) \geq t \mid 1 - \phi_t < (1-\nu)t/n \cap \mathcal{H}_t] \leq \mathbb{P}[\text{Bin}(n, (1-\nu)t/n) \geq t \mid \mathcal{H}_t].$$

²We will use the notation “with high probability” and denote as **whp**, meaning with probability $1 - o(1)$, as the number of nodes $n \rightarrow \infty$.

Taking the expectation with respect to the history \mathcal{H}_t in (25) yields

$$\mathbb{P}[T(v_0) > t] \leq \mathbb{P}[\text{Bin}(n, (1-\nu)t/n) \geq t] + \mathbb{P}[1 - \phi_{t-1} \geq (1-\nu)t/n].$$

For $t = K_0 \log n$, where K_0 is a constant large enough and independent on the initial node v_0 , the Chernoff bound ensures that $\mathbb{P}[\text{Bin}(n, (1-\nu)t/n) \geq t] = o(1/n)$. To bound $\mathbb{P}[1 - \phi_{t-1} \geq (1-\nu)t/n \mid \mathcal{H}_t]$, use (19) to obtain

$$\begin{aligned} \{1 - \phi_{t-1} \geq (1-\nu)t/n\} &= \left\{ \prod_{w \in W} q_w^{\mathbb{I}(\Gamma_w \leq t)} \leq 1 - \frac{(1-\nu)t}{n} \right\} \\ &= \left\{ \sum_{w \in W} \log \left(\frac{1}{1-p_w} \right) \mathbb{I}(\Gamma_w \leq t) \geq -\log \left(1 - \frac{(1-\nu)t}{n} \right) \right\}. \end{aligned}$$

Linearize $-\log(1 - (1-\nu)t/n) = (1-\nu)t/n + o(t/n)$ and define the bounded auxiliary random variables $X_{t,w} = n \log(1/(1-p_w)) \mathbb{I}(\Gamma_w \leq t)$. Direct calculations reveal that

$$\begin{aligned} \mathbb{E}[X_{t,w}] &= n \log \left(\frac{1}{1-p_w} \right) (1 - q_w^t) = n(p_w + \phi(p_w)) (1 - (1-p_w)^t) \\ &= n(p_w + \phi(p_w)) (tp_w + \phi(tp_w)) = ntp_w^2 + \phi(ntp_w^2), \end{aligned} \quad (26)$$

which implies

$$\sum_{w \in W} \mathbb{E}[X_{t,w}] = nt \sum_{w \in W} p_w^2 + \phi \left(nt \sum_{w \in W} p_w^2 \right). \quad (27)$$

Thus under the stated condition that

$$n \sum_{w \in W} p_w^2 \leq c < 1,$$

it follows that $0 < (1-c)t \leq t - \sum_{w \in W} \mathbb{E}[X_{t,w}]$. In light of Bernstein's inequality [5], we bound

$$\begin{aligned} \mathbb{P}[1 - \phi_{t-1} \geq (1-\nu)t/n] &= \mathbb{P} \left[\sum_{w \in W} X_{t,w} \geq (1-\nu)t \right] \leq \mathbb{P} \left[\sum_{w \in W} (X_{t,w} - \mathbb{E}[X_{t,w}]) \geq (1-\nu-c)t \right] \\ &\leq \exp \left(- \frac{\frac{3}{2}((1-\nu-c)t)^2}{3 \sum_{w \in W} \text{Var}[X_{t,w}] + nt \max_w \{p_w\} (1 + \phi(1))} \right). \end{aligned} \quad (28)$$

Since

$$\begin{aligned} \mathbb{E}[X_{t,w}^2] &= \left(n \log \left(\frac{1}{1-p_w} \right) \right)^2 (1 - q_w^t) = n^2 (p_w + \phi(p_w))^2 (1 - (1-p_w)^t) \\ &= n^2 (p_w^2 + \phi(p_w^2)) (tp_w + \phi(tp_w)) = n^2 tp_w^3 + \phi \left(n^2 t \sum_{w \in W} p_w^3 \right), \end{aligned} \quad (29)$$

it follows that for some large constant $K_1 > 0$

$$\sum_{w \in W} \text{Var}[X_{t,w}] \leq \sum_{w \in W} \mathbb{E}[X_{t,w}^2] = n^2 t \sum_{w \in W} p_w^3 + \phi \left(n^2 t \sum_{w \in W} p_w^3 \right) \leq K_1 t.$$

Finally, the assumption of the theorem implies that there exists constant $K_2 > 0$ such that

$$n \max_{w \in W} p_w \leq K_2.$$

Substituting these bounds into (28) yields

$$\mathbb{P}[1 - \phi_{t-1} \geq (1-\nu)t/n] \leq \exp \left(- \frac{3(1-\nu-c)^2}{2(3K_1 + K_2)} t \right),$$

and taking $\nu \in (0, 1 - c)$ and $t = K_3 \log n$ for some constant K_3 large enough and not depending on the initial node v_0 , we conclude that $\mathbb{P}[1 - \phi_{t-1} \geq (1 - \nu)t/n] = o(n^{-1})$, which in turn implies that taking constant $K_4 = \max\{K_0, K_3\}$, ensures that

$$\mathbb{P}[T(v_0) > K_4 \log n] = o(1/n)$$

for any initial node v_0 . Finally, a union bound over the n possible starting values v_0 implies that

$$\mathbb{P}[\max_{v_0 \in V} T(v_0) > K_4 \log n] \leq n o(n^{-1}) = o(1),$$

which implies that all connected components in the random intersection are of size $O(\log n)$, **whp**. ■

Remarks. We now consider the conditions of the theorem. From the Cauchy-Schwarz inequality, we obtain $\left(\sum_{w \in W} p_w^3\right) \left(\sum_{w \in W} p_w\right) \geq \left(\sum_{w \in W} p_w^2\right)^2$. Moreover, given that $\sum_{w \in W} p_w^3 = O(1/n^2)$ and $p_w = O(1/n)$, it follows $\sum_{w \in W} p_w^2 = \Omega(\sqrt{m/n^3})$. Hence, for $\sum_{w \in W} p_w^2 = c/n$, when $c < 1$, it follows $m = \Omega(n)$, which is consistent with the results in [4] on the non-existence of a giant component in a uniform RIG.

5.2 Supercritical regime

We now turn to the study of the supercritical regime in which $\lim_{n \rightarrow \infty} n \sum_{w \in W} p_w^2 = c > 1$.

Theorem 6 *Let*

$$\sum_{w \in W} p_w^3 = o\left(\frac{\log n}{n^2}\right) \quad \text{and} \quad p_w = o\left(\frac{\log n}{n}\right), \quad \text{for all } w.$$

*For any constant $c > 1$, if $\sum_{w \in W} p_w^2 \geq c/n$, then **whp** there exists a unique largest component in $G(n, m, \mathbf{p})$, of order $\Theta(n)$. Moreover, the size of the giant component is given by $n\zeta_c(1 + o(1))$, where ζ_c is the solution in $(0, 1)$ of the equation $1 - e^{-c\zeta} = \zeta$, while all other components are of size $O(\log n)$.*

Remarks. The conditions on p_w and $\sum_{w \in W} p_w^3$ are weaker than ones in the case of the sub-critical regime.

The proof proceeds as follows. The first step is to bound, both from above and below, the value $1 - \prod_{\tau=0}^{t-1} (1 - r_\tau)$ that governs the behavior the branching process $\{Y_t\}_{t \geq 0}$, see Lemma 3. With the lower bound, we show the emergence with high probability of at least one giant component of size $\Theta(n)$. We use the upper bound to prove uniqueness of the giant component. Technically, we make use of these bounds to compare our branching process to branching processes arising in the study of Erdős-Rényi random graphs.

Proof We start by bounding $1 - \prod_{\tau=0}^{t-1} (1 - r_\tau)$. The upper bounds $\sum_{\tau=0}^{t-1} r_\tau$ has been previously established in (24). For the lower bound, we apply Jensen's inequality to the function $\log(1 - x)$ to get

$$\begin{aligned} \log \prod_{\tau=0}^{t-1} (1 - r_\tau) &= \sum_{\tau=0}^{t-1} \log(1 - r_\tau) = \sum_{\tau=0}^{t-1} \log \left(1 - (\phi_{\tau-1} - \phi_\tau)\right) \\ &\leq t \log \left(1 - \frac{1}{t} \sum_{\tau=0}^{t-1} (\phi_{\tau-1} - \phi_\tau)\right) = t \log \left(1 - \frac{1 - \phi_{t-1}}{t}\right). \end{aligned} \quad (30)$$

In light of (19), ϕ_t is decreasing in t , and hence

$$1 - \left(1 - \frac{1 - \phi_0}{t}\right)^t \leq 1 - \left(1 - \frac{1 - \phi_{t-1}}{t}\right)^t \leq 1 - \prod_{\tau=0}^{t-1} (1 - r_\tau) \leq \sum_{\tau=0}^{t-1} r_\tau = 1 - \phi_{t-1}. \quad (31)$$

To further bound $1 - \left(1 - \frac{1-\phi_0}{t}\right)^t$, consider the function $f_t(x) = 1 - (1 - x/t)^t$ for x in a neighborhood of the origin and $t \geq 1$. For any fixed x , $f_t(x)$ decreases to $1 - e^{-x}$ as t tends to infinity. The latter function is concave, and hence for all $x \leq \varepsilon$,

$$\frac{1 - e^{-\varepsilon}}{\varepsilon} x \leq f_t(x).$$

Note that $(1 - e^{-\varepsilon})/\varepsilon$ can be made arbitrary close to one by taking ε small enough. Furthermore, $f_t(x)$ is increasing in x for fixed t . From (19), $1 - \phi_0 \leq 1 - \phi_t$, hence $1 - (1 - \frac{1-\phi_0}{t})^t \leq 1 - (1 - \frac{1-\phi_{t-1}}{t})^t$. Looking closer at $1 - \phi_0$, from (22) and (21), by using Chebyshev inequality, with $\sum_{w \in W} p_w^2 = c/n$, it follows that ϕ_0 is concentrated around its mean $\mathbb{E}[\phi_0] = c/n$. That is, for any constant $\delta > 0$, $\phi_0 \in ((1 - \delta)c/n, (1 + \delta)c/n)$, with probability $1 - o(1/n)$. We conclude that for any $\delta > 0$ there is $\epsilon > 0$ such that $(c - \delta)\frac{1-e^{-\epsilon}}{\epsilon} > 1$, since constant $c > 1$. Moreover, since $\lim_{\epsilon \rightarrow 0} \frac{1-e^{-\epsilon}}{\epsilon} = 1$, by choosing ϵ sufficiently small, $\frac{1-e^{-\epsilon}}{\epsilon}$ can be arbitrarily close to 1. It follows that $1 - \prod_{\tau=0}^{t-1} (1 - r_\tau) > c'/n$, for some constant $c > c' > 1$ arbitrarily close to c . Hence, the branching process on RIG is stochastically lower bounded by the $\text{Bin}(n - 1, c'/n)$, which stochastically dominates a branching process on $G_{n, c'/n}$. Because $c' > 1$, there exists **whp** a giant component of size $\Theta(n)$ in $G_{n, c'/n}$. This implies that the stopping of the branching process associated to $G_{n, c'/n}$ is $\Theta(n)$ with high probability, and so is the stopping time T_v for some $v \in V$, which implies that there is a giant component in a general RIG, **whp**.

Let us look closer at the size of that giant component. From the representation (19) for ϕ_{t-1} , consider the previously introduced random variables $X_{t,w} = n \log(1/(1 - p_w)) \mathbb{I}(\Gamma_w \leq t)$. Similarly, as in the proof of the Theorem 5, it follows that under the conditions of the theorem there is a positive constant $\delta > 0$ such that $\sum_w X_{t,w}$ is concentrated within $(1 \pm \delta) \sum_w \mathbb{E}[X_{t,w}] = (1 \pm \delta)c/n$, with probability $1 - o(1)$. Hence, there exists $p^+ = c^+/n$, for some constant $c^+ > c > 1$, such that $1 - \phi_{t-1} \leq 1 - (1 - p^+)^t$, which is equivalent to $-\log \phi_{t-1} \leq t \log(1 - p^+) = tp^+ + o(tp^+) = tc^+/n + o(t/n)$. Similarly, the concentration of ϕ_{t-1} implies that there exists $p^- = c^-/n$, with $c > c^- > 1$, such that $1 - (1 - p^-)^t \leq 1 - (1 - (1 - \phi_{t-1})/t)^t$, which implies that $-\log \phi_{t-1} \geq t \log(1 - p^-) = tp^- + o(tp^-) = tc^-/n + o(t/n)$. Combining the upper and lower bound, we conclude that with probability $1 - o(1)$, the rate of the branching process on RIG is bracketed by

$$1 - (1 - p^-)^t \leq 1 - \prod_{\tau=0}^{t-1} (1 - r_\tau) \leq 1 - (1 - p^+)^t. \quad (32)$$

The stochastic dominance of the Binomial distribution together with (32), implies

$$\begin{aligned} \mathbb{P}\left[\text{Bin}\left(n - 1, 1 - (1 - p^-)^t\right) \geq t\right] &\leq \mathbb{P}\left[\text{Bin}\left(n - 1, 1 - \prod_{\tau=0}^{t-1} (1 - r_\tau)\right) \geq t\right] \\ &\leq \mathbb{P}\left[\text{Bin}\left(n - 1, 1 - (1 - p^+)^t\right) \geq t\right]. \end{aligned} \quad (33)$$

In light of (32), the branching process $\{Y_t\}_{t \geq 0}$ associated to a RIG is stochastically bounded from below and from above by the branching processes associated to G_{n, p^-} and G_{n, p^+} , respectively (for the analysis on an Erdős-Rényi graph, see [2]). Since both $c^-, c^+ > 1$, there exist giant components in both G_{n, p^-} and G_{n, p^+} , **whp**.

In [22], it has been shown that the giant components in $G_{n, \lambda/n}$, for $\lambda > 1$, is unique and of size $\approx n\zeta_\lambda$, where ζ_λ is the unique solution from $(0, 1)$ of the equation

$$1 - e^{-\lambda\zeta} = \zeta. \quad (34)$$

Moreover, the size of the giant component in $G_{n, \lambda/n}$ satisfies the central limit theorem

$$\frac{\max_v \{|C(v)|\} - \zeta_\lambda n}{\sqrt{n}} \stackrel{d}{=} \mathcal{N}\left(0, \frac{\zeta_\lambda(1 - \zeta_\lambda)}{(1 - \lambda + \lambda\zeta_\lambda)^2}\right). \quad (35)$$

From the definition of the stopping time, see (23), and since (33) and (35), it follows there is a giant component in a RIG, of size, at least, $n\zeta_\lambda(1 - \phi(1))$, **whp**. Furthermore, the stopping times of the branching processes associated to G_{n,p^-} and G_{n,p^+} are approximately ζn , where ζ satisfy (34), with $\lambda^- = np^-$ and $\lambda^+ = np^+$, respectively. These two stopping times are close to one another, which follows from analyzing the function $F(\zeta, c) = 1 - \zeta - e^{-c\zeta}$, where (ζ, c) is the solution of $F(\zeta, c) = 0$, for given c . Since all partial derivatives of $F(\zeta, c)$ are continuous and bounded, the stopping times of the branching processes defined from G_{n,p^-} , G_{n,p^+} are ‘close’ to the solution of (34), for $\lambda = c$. From (33), the stopping time of a RIG is bounded by the stopping times on G_{n,p^-} , G_{n,p^+} .

We conclude by proving that **whp**, the giant component of a RIG is unique by adapting the arguments in [2] to our setting. Let us assume that there are at least two giant components in a RIG, with the sets of nodes $V_1, V_2 \subset V$. Let us create a new, independent ‘sprinkling’ $\widehat{\text{RIG}}$ on the top of our RIG, with the same sets of nodes and attributes, while $\hat{p}_w = p_w^\gamma$, for $\gamma > 1$ to be defined later. Now, our object of interest is $\text{RIG}_{new} = \text{RIG} \cup \widehat{\text{RIG}}$. Let us consider all $\Theta(n^2)$ pairs $\{v_1, v_2\}$, where $v_1 \in V_1, v_2 \in V_2$, which are independent in $\widehat{\text{RIG}}$, (but not in RIG), hence the probability that two nodes $v_1, v_2 \in V$ are connected in $\widehat{\text{RIG}}$ is given by

$$1 - \prod_w (1 - \hat{p}_w^2) = 1 - \prod_w (1 - p_w^{2\gamma}) = \sum_w p_w^{2\gamma} + \phi\left(\sum_w p_w^{2\gamma}\right), \quad (36)$$

which is true, since $\gamma > 1$ and $p_w = O(1/n)$ for any w . Given that $\sum_w p_w^2 = c/n$, we choose $\gamma > 1$ so that $\sum_w p_w^{2\gamma} = \omega(1/n^2)$. Now, by the Markov inequality, **whp** there is a pair $\{v_1, v_2\}$ such that v_1 is connected to v_2 in $\widehat{\text{RIG}}$, implying that V_1, V_2 are connected, **whp**, forming one connected component within RIG_{new} . From the previous analysis, it follows that this component is of size at least $2n\zeta_\lambda(1 - \delta)$ for any small constant $\delta > 0$. On the other hand, the probabilities p_w^{new} in RIG_{new} satisfy

$$p_w^{new} = 1 - (1 - p_w)(1 - \hat{p}_w) = p_w + \hat{p}_w(1 - p_w) = p_w + p_w^\gamma(1 - p_w) = p_w(1 + \phi(1)),$$

which is again true, since $\gamma > 1$ and $p_w = O(1/n)$ for any w . Thus,

$$\sum_{w \in W} (p_w^{new})^2 = \sum_{w \in W} p_w^2 + \Theta\left(\sum_{w \in W} p_w^{1+\gamma}(1 - p_w)\right) = \sum_{w \in W} p_w^2(1 + \phi(1)) = c/n + o(1/n). \quad (37)$$

Given that the stopping time on RIG is bounded by the stopping times on G_{n,p^-} , G_{n,p^+} , and from its continuity, it follows that the giant component in RIG_{new} cannot be of size $2n\zeta_\lambda(1 - \delta)$, which is a contradiction. Thus, there is only one giant component in RIG, of size given by $n\zeta_c(1 + \phi(1))$, where ζ_c satisfies (34), for $\lambda = c$. Moreover, knowing behavior of $G_{n,p}$, from (33), it follows that all other components are of size $O(\log n)$. ■

6 Conclusion

The analysis of random models for bipartite graphs is important for the study of social networks, or any network formed by associating nodes with shared attributes. In the random intersection graph (RIG) model, nodes have certain attributes with fixed probabilities. In this paper, we have considered the general RIG model, where these probabilities are represented by a set of probabilities $\mathbf{p} = \{p_w\}_{w \in W}$, where p_w denotes the probability that a node is attached to the attribute w .

We have analyzed the evolution of components in general RIGs, giving conditions for existence and uniqueness of the giant component. We have done so by generalizing the branching process argument used to study the birth of the giant component in Erdős-Rényi graphs. We have considered a dependent, inhomogeneous Galton-Watson process, where the number of offspring follows a binomial distribution with a different number of nodes and different rate at each step during the evolution. The analysis of such a process is complicated by the dependence on its history, dictated by the structure of general RIGs. We have shown that in spite of this difficulty, it is possible to give stochastic bounds on the branching process, and that under certain conditions the giant component appears at the threshold $n \sum_{w \in W} p_w^2 = 1$, with probability tending to one, as the number of nodes tends to infinity.

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